

# On Koksma-Hlawka inequality

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## Abstract

The classical Koksma Hlawka inequality does not apply to functions with simple discontinuities. Here we state a Koksma Hlawka type inequality which applies to piecewise smooth functions  $f\chi_\Omega$ , with  $f$  smooth and  $\Omega$  a Borel subset of  $[0, 1]^d$ :

$$\left| N^{-1} \sum_{j=1}^N (f\chi_\Omega)(x_j) - \int_\Omega f(x) dx \right| \leq \mathcal{D}(\Omega, x_j) \mathcal{V}(f),$$

where  $\mathcal{D}(\Omega, x_j)$  is the discrepancy

$$\mathcal{D}(\Omega, x_j) = 2^d \sup_{I \subseteq [0,1]^d} \left\{ \left| N^{-1} \sum_{j=1}^N \chi_{\Omega \cap I}(x_j) - |\Omega \cap I| \right| \right\},$$

the supremum is over all  $d$ -dimensional intervals, and  $\mathcal{V}(f)$  is the total variation

$$\mathcal{V}(f) = \sum_{\alpha \in \{0,1\}^d} 2^{d-|\alpha|} \int_{[0,1]^d} \left| \left( \frac{\partial}{\partial x} \right)^\alpha f(x) \right| dx.$$

We state similar results with variation and discrepancy measured by  $L^p$  and  $L^q$  norms,  $1/p + 1/q = 1$ , and we also give extensions to compact manifolds.

## 1 Introduction

Koksma's inequality is a neat bound for the error in a numerical integration:

$$\left| N^{-1} \sum_{j=1}^N f(x_j) - \int_0^1 f(x) dx \right| \leq \mathcal{D}(x_j) \mathcal{V}(f).$$

In this inequality  $\mathcal{D}(x_j)$  is the discrepancy of the points  $0 \leq x_j \leq 1$  and  $\mathcal{V}(f)$  is the total variation of the function  $f$ ,

$$\mathcal{D}(x_j) = \sup_{0 \leq t \leq 1} \left\{ \left| N^{-1} \sum_{j=1}^N \chi_{[0,t]}(x_j) - t \right| \right\},$$

$$\mathcal{V}(f) = \sup_{0=y_0 < y_1 < y_2 < \dots < y_n=1} \left\{ \sum_{k=1}^n |f(y_k) - f(y_{k-1})| \right\}.$$

We may say that Koksma's inequality is a simple machine which turns the discrepancy for a small family of functions, characteristic functions of intervals, into the discrepancy for a larger family, functions of bounded variation. The extension to several variables is a more delicate problem, yet it is of some relevance in numerical analysis. See e.g. [8], [9], [10], [11], [12], [15]. A classical approach starts with the definitions of Vitali and Hardy Krause variations. For a function  $f$  on  $[0, 1]^d$  and for every  $d$  dimensional interval  $I$  in  $[0, 1]^d$  with edges parallel to the axes, let  $\Delta(f, I)$  be an alternating sum of the values of  $f$  at the vertices of  $I$ . The Vitali variation is

$$V(f) = \sup_R \left\{ \sum_{I \in R} |\Delta(f, I)| \right\},$$

where the supremum is over all finite partitions  $R$  of  $[0, 1]^d$  in  $d$  dimensional intervals  $I$ . The Hardy Krause variation is

$$\mathcal{V}(f) = \sum_k V_k(f),$$

where the sum is over the Vitali variations  $V_k(f)$  of the restrictions of  $f$  to all faces of all dimensions of  $[0, 1]^d$ . The discrepancy of a finite point set  $\{x_j\}_{j=1}^N$  in  $[0, 1]^d$  is defined by

$$\mathcal{D}(x_j) = \sup_I \left\{ \left| N^{-1} \sum_{j=1}^N \chi_I(x_j) - |I| \right| \right\},$$

where  $I$  is an interval of the form  $[0, t_1] \times [0, t_2] \times \dots \times [0, t_d]$  with  $0 \leq t_k \leq 1$ , and  $|I| = t_1 t_2 \dots t_d$  is its measure. The classical Koksma Hlawka inequality states that if  $f$  has bounded Hardy Krause variation, then

$$\left| N^{-1} \sum_{j=1}^N f(x_j) - \int_{[0,1]^d} f(x) dx \right| \leq \mathcal{D}(x_j) \mathcal{V}(f).$$

The assumptions required in the one dimensional Koksma inequality are satisfied by many familiar functions and are usually easy to verify. On the contrary, the Hardy Krause condition in the Koksma Hlawka inequality seems to be rather strict. It works well for smooth functions, but it cannot be applied to most functions with simple discontinuities. For example, the characteristic function of a convex polyhedron has bounded Hardy Krause variation only if the polyhedron is a  $d$  dimensional interval. For this and other reasons, several variants of the Koksma Hlawka inequality have been proposed. In particular, in [7] the small family consists of characteristic functions of convex sets and the large family is given by functions with super level sets which are differences of finite unions of convex sets. See also [6, p.162], [11], [14]. Finally, a general and systematic approach to Koksma Hlawka inequalities is via reproducing kernel Hilbert spaces. See e.g. [1] and [8]. However, in some of these approaches the geometric meaning of the discrepancy is somehow hidden. The aim of this paper is to state some Koksma Hlawka inequalities with explicit geometric discrepancies, and which apply to piecewise smooth functions, that is smooth functions  $f$  restricted to arbitrary Borel sets  $\Omega$ . In one version of this inequality the error in the numerical integration of  $f\chi_\Omega$  is controlled by a variation of  $f$  defined in terms of derivatives, times the discrepancy of the intersection of  $\Omega$  with translates of intervals  $I$  with edges parallel to the axes. In another version the discrepancy is with respect to the intersection of  $\Omega$  with cubes, and in a further version the discrepancy is with respect to the intersection of  $\Omega$  with balls. These results are first stated and proved when the underlying space is a torus, then they are extended to compact manifolds, in particular spheres.

## 2 Koksma Hlawka inequalities on a torus

In what follows we are going to consider functions, measures, distributions, on the torus  $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d = [0, 1)^d$ , that is functions, measures, distributions on  $\mathbb{R}^d$  which are  $\mathbb{Z}^d$  periodic.

**Theorem 1** *If  $f$  and  $\mu$  are an integrable function and a finite measure on  $\mathbb{T}^d$  respectively, if  $\Omega$  is a bounded Borel subset of  $\mathbb{R}^d$ , and if  $1 \leq p, q \leq +\infty$  with  $1/p + 1/q = 1$ , then*

$$\left| \int_{\Omega} f(x) d\mu(x) \right| \leq \mathcal{D}_q(\Omega, \mu) \mathcal{V}_p(f),$$

where  $\mathcal{D}_q(\Omega, \mu)$  is the  $L^q$  discrepancy

$$\mathcal{D}_q(\Omega, \mu) = \int_{[0,1]^d} \left\{ \int_{\mathbb{T}^d} \left| \sum_{n \in \mathbb{Z}^d} \mu((x + n - I(t)) \cap \Omega) \right|^q dx \right\}^{1/q} dt,$$

with  $I(t) = [0, t_1] \times [0, t_2] \times \dots \times [0, t_d]$ ,  $0 \leq t_k \leq 1$ , and  $\mathcal{V}_p(f)$  is the  $L^p$  total variation

$$\mathcal{V}_p(f) = \sum_{\alpha \in \{0,1\}^d} 2^{d-|\alpha|} \left\{ \int_{\mathbb{T}^d} \left| \left( \frac{\partial}{\partial x} \right)^\alpha f(x) \right|^p dx \right\}^{1/p}.$$

The variation of the function and the discrepancy of the measure in the statement of the theorem increase with  $p$  and  $q$ , hence a gain in  $q$  corresponds to a loss in  $p$ . When  $\Omega$  is contained in  $[0, 1]^d$  the  $L^\infty$  discrepancy is dominated as follows:

$$\mathcal{D}_\infty(\Omega, \mu) \leq 2^d \sup_{I \subseteq [0,1]^d} \{|\mu(I \cap \Omega)|\}.$$

This reflects the difference between the discrepancy in a torus and the one in a cube, and it is due to the fact that an interval in  $\mathbb{T}^d$  can be split into at most  $2^d$  intervals in  $[0, 1]^d$ . In the above formulas with  $p = 1$  the derivatives can be measures, and in this case the norms  $\int_{\mathbb{T}^d} |(\partial/\partial x)^\alpha f(x)| dx$  denote the total variations. Observe that if  $p = 1$  then less than  $d$  integrable derivatives are not enough to guarantee the boundedness of the functions. Hence, if the measure  $\mu$  is concentrated on the singularities of the function  $f$ , the integral  $\int_\Omega f(x) d\mu(x)$  may be not defined. In the classical Koksma Hlawka inequality, and in most of discrepancy theory, the measure  $\mu$  is the difference between masses  $\delta_{x_j}$  concentrated at the points  $x_j$  and the uniformly distributed measure  $dx$ :

$$d\mu = N^{-1} \sum_{j=1}^N \delta_{x_j} - dx.$$

For this measure and when  $p = 1$  and  $\Omega = [0, 1]^d$  the above theorem is essentially equivalent to the classical Koksma Hlawka inequality with respect to the Hardy Krause variation. The proof of the theorem can be split into a sequence of easy lemmas. The first one can be seen as a Fourier analog of a multidimensional integration by parts in [15]. See also the examples in [1].

**Lemma 2** *Let  $\varphi$  be a non vanishing complex sequence on  $\mathbb{Z}^d$ , and assume that both  $\varphi$  and  $1/\varphi$  have tempered growth in  $\mathbb{Z}^d$ . Also let  $f$  be an integrable function on  $\mathbb{T}^d$ . Define*

$$g(x) = \sum_{n \in \mathbb{Z}^d} \overline{\varphi(n)^{-1}} e^{2\pi i n \cdot x},$$

$$\mathfrak{D}f(x) = \sum_{n \in \mathbb{Z}^d} \varphi(n) \widehat{f}(n) e^{2\pi i n \cdot x}.$$

*Finally, let  $\mu$  be a finite measure on  $\mathbb{T}^d$ . Then, the following identity holds:*

$$\int_{\mathbb{T}^d} f(x) \overline{d\mu(x)} = \int_{\mathbb{T}^d} \mathfrak{D}f(x) \overline{g * \mu(x)} dx.$$

**Proof.** We are using the notation  $\widehat{f}(n) = \int_{\mathbb{T}^d} f(x) e^{-2\pi i n \cdot x} dx$  for the Fourier transform and  $g * \mu(x) = \int_{\mathbb{T}^d} g(x - y) d\mu(y)$  for the convolution, and we are applying these operators also to distributions. In particular, the assumptions on the growth of  $\varphi$  and  $1/\varphi$  guarantee that both  $\mathfrak{D}f$  and  $g$  are well defined as tempered distributions. Moreover

$$\begin{aligned} \int_{\mathbb{T}^d} f(x) \overline{d\mu(x)} &= \sum_{n \in \mathbb{Z}^d} \widehat{f}(n) \overline{\widehat{\mu}(n)} \\ &= \sum_{n \in \mathbb{Z}^d} \left( \varphi(n) \widehat{f}(n) \right) \left( \varphi(n)^{-1} \overline{\widehat{\mu}(n)} \right) = \int_{\mathbb{T}^d} \mathfrak{D}f(x) \overline{g * \mu(x)} dx. \end{aligned}$$

■

Suitable choices of  $\varphi$  and  $\mu$  will make the above abstract lemma more concrete and interesting. In particular,  $\varphi$  will be the Fourier transform of a differential integral operator and  $1/\varphi$  the Fourier transform of a superposition of characteristic functions.

**Lemma 3** *Let the function  $h$  on  $\mathbb{R}^d$  be the superposition of all intervals  $I(t) = [0, t_1] \times [0, t_2] \times \dots \times [0, t_d]$  with  $0 \leq t_k \leq 1$ , and let  $g(x)$  be the  $\mathbb{Z}^d$  periodization of  $h$ ,*

$$h(x) = \int_{[0,1]^d} \chi_{I(t)}(x) dt, \quad g(x) = \sum_{n \in \mathbb{Z}^d} h(x + n).$$

*Then the function  $g$  has Fourier expansion*

$$g(x) = \sum_{n \in \mathbb{Z}^d} \left( \prod_{k=1}^d (2\delta(n_k) + 2\pi i n_k)^{-1} \right) e^{2\pi i n \cdot x},$$

*where  $\delta(0) = 1$  and  $\delta(j) = 0$  for  $j \neq 0$ .*

**Proof.** Observe that

$$h(x) = \prod_{k=1}^d \int_0^1 \chi_{[0,t_k]}(x_k) dt_k = \prod_{k=1}^d (1 - x_k) \chi_{[0,1]}(x_k).$$

Then compute the Fourier coefficients,

$$\begin{aligned} \widehat{g}(n) &= \int_{\mathbb{T}^d} g(x) e^{-2\pi i n \cdot x} dx \\ &= \prod_{k=1}^d \left( \int_0^1 (1 - x_k) e^{-2\pi i n_k x_k} dx_k \right) = \prod_{k=1}^d (2\delta(n_k) + 2\pi i n_k)^{-1}. \end{aligned}$$

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**Lemma 4** *If  $f$  is a smooth function on  $\mathbb{T}^d$ , then*

$$\begin{aligned} \mathfrak{D}f(x) &= \sum_{n \in \mathbb{Z}^d} \left( \prod_{k=1}^d (2\delta(n_k) - 2\pi i n_k) \right) \widehat{f}(n) \exp(2\pi i n x) \\ &= \sum_{\alpha, \beta \in \{0,1\}^d, \alpha + \beta = (1, \dots, 1)} (-1)^{|\alpha|} 2^{|\beta|} \int_{[0,1]^{|\beta|}} \left( \frac{\partial}{\partial x} \right)^\alpha f(x + y^\beta) dy^\beta. \end{aligned}$$

We are using the notation  $(\partial/\partial x)^\alpha = (\partial/\partial x_1)^{\alpha_1} \dots (\partial/\partial x_d)^{\alpha_d}$  and  $y^\beta = \sum_{j=1}^d y_j^{\beta_j} e_j$ , where  $\{e_j\}_{j=1}^d$  is the canonical basis of  $\mathbb{R}^d$  and  $dy^\beta = dy_1^{\beta_1} \dots dy_d^{\beta_d}$ .

**Proof.** In order to avoid a heavy notation and to make the proof transparent, we consider only the case  $d = 2$  and we write  $(x, y)$  and  $(m, n)$  in place of  $x$  and  $n$ .

$$\begin{aligned} \mathfrak{D}f(x, y) &= \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} (2\delta(m) - 2\pi i m) (2\delta(n) - 2\pi i n) \widehat{f}(m, n) e^{2\pi i (mx + ny)} \\ &= 4\widehat{f}(0, 0) - 2 \sum_{m \in \mathbb{Z}} 2\pi i m \widehat{f}(m, 0) e^{2\pi i m x} - 2 \sum_{n \in \mathbb{Z}} 2\pi i n \widehat{f}(0, n) e^{2\pi i n y} \\ &\quad + \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} (2\pi i m) (2\pi i n) \widehat{f}(m, n) e^{2\pi i (mx + ny)} \\ &= 4 \int_0^1 \int_0^1 f(x, y) dx dy - 2 \int_0^1 \frac{\partial f}{\partial x}(x, y) dy - 2 \int_0^1 \frac{\partial f}{\partial y}(x, y) dx + \frac{\partial^2 f}{\partial x \partial y}(x, y). \end{aligned}$$

■

The following lemma is nothing but a restatement of the theorem.

**Lemma 5** *If  $1 \leq p, q \leq +\infty$  and  $1/p + 1/q = 1$ , then*

$$\begin{aligned} & \left| \int_{\Omega} f(x) \overline{d\mu(x)} \right| \\ & \leq \left( \int_{[0,1]^d} \left\{ \int_{\mathbb{T}^d} \left| \sum_{n \in \mathbb{Z}^d} \mu((x+n-I(t)) \cap \Omega) \right|^q dx \right\}^{1/q} dt \right) \\ & \quad \times \left( \sum_{\alpha \in \{0,1\}^d} 2^{d-|\alpha|} \left\{ \int_{\mathbb{T}^d} \left| \left( \frac{\partial}{\partial x} \right)^\alpha f(x) \right|^p dx \right\}^{1/p} \right). \end{aligned}$$

**Proof.** We have to integrate a periodic function  $f$  against a periodic measure  $\mu$  over an arbitrary non periodic Borel set  $\Omega$  in  $\mathbb{R}^d$ . By Lemma 2 applied to the periodization  $\nu$  of the measure  $\chi_{\Omega}\mu$ , and by Hölder inequality,

$$\begin{aligned} \left| \int_{\Omega} f(x) \overline{d\mu(x)} \right| &= \left| \int_{\mathbb{T}^d} f(x) \left( \sum_{n \in \mathbb{Z}^d} \chi_{\Omega}(x+n) \right) \overline{d\mu(x)} \right| \\ &= \left| \int_{\mathbb{T}^d} f(x) \overline{d\nu(x)} \right| \leq \|\mathfrak{D}f\|_{L^p(\mathbb{T}^d)} \|g * \nu\|_{L^q(\mathbb{T}^d)}. \end{aligned}$$

The estimate for  $\|\mathfrak{D}f\|_{L^p(\mathbb{T}^d)}$  follows from Lemma 4,

$$\left\{ \int_{\mathbb{T}^d} |\mathfrak{D}f(x)|^p dx \right\}^{1/p} \leq \sum_{\alpha \in \{0,1\}^d} 2^{d-|\alpha|} \left\{ \int_{\mathbb{T}^d} \left| \left( \frac{\partial}{\partial x} \right)^\alpha f(x) \right|^p dx \right\}^{1/p}.$$

The estimate for  $\|g * \nu\|_{L^q(\mathbb{T}^d)}$  follows from Lemma 3,

$$\begin{aligned} & \left\{ \int_{\mathbb{T}^d} |g * \nu(x)|^q dx \right\}^{1/q} \\ &= \left\{ \int_{\mathbb{T}^d} \left| \int_{\mathbb{T}^d} \left( \sum_{m \in \mathbb{Z}^d} \int_{[0,1]^d} \chi_{I(t)}(x-y+m) dt \right) \left( \sum_{n \in \mathbb{Z}^d} \chi_{\Omega}(y+n) d\mu(y) \right) dx \right|^q dx \right\}^{1/q} \\ &= \left\{ \int_{\mathbb{T}^d} \left| \int_{[0,1]^d} \left( \sum_{n \in \mathbb{Z}^d} \int_{\mathbb{R}^d} \chi_{I(t)}(x+n-z) \chi_{\Omega}(z) d\mu(z) \right) dt \right|^q dx \right\}^{1/q} \\ &= \left\{ \int_{\mathbb{T}^d} \left| \int_{[0,1]^d} \left( \sum_{n \in \mathbb{Z}^d} \mu((x+n-I(t)) \cap \Omega) \right) dt \right|^q dx \right\}^{1/q} \\ &\leq \int_{[0,1]^d} \left\{ \int_{\mathbb{T}^d} \left| \sum_{n \in \mathbb{Z}^d} \mu((x+n-I(t)) \cap \Omega) \right|^q dx \right\}^{1/q} dt. \end{aligned}$$

■

The following are a few applications.

**Corollary 6** *Let  $\gamma = 4$  if  $d = 2$ ,  $\gamma = 3/2$  if  $d = 3$ ,  $\gamma = 2/(d + 1)$  if  $d \geq 4$ . Then there is a constant  $c$  depending only on the dimension such that for every  $N \geq 2$  there exists a finite sequence of points  $\{x_j\}_{j=1}^N$  in  $[0, 1]^d$  with the following property: For every convex set  $\Omega$  contained in  $[0, 1]^d$  and for every smooth function  $f$  on  $\mathbb{T}^d$ ,*

$$\begin{aligned} & \left| N^{-1} \sum_{j=1}^N (f \chi_{\Omega})(x_j) - \int_{\Omega} f(x) dx \right| \\ & \leq c N^{-2/(d+1)} \log^{\gamma}(N) \sum_{\alpha \in \{0,1\}^d} \int_{[0,1]^d} \left| \left( \frac{\partial}{\partial x} \right)^{\alpha} f(x) \right| dx. \end{aligned}$$

**Proof.** Apply the theorem to the measure  $d\mu = N^{-1} \sum_{j=1}^N \delta_{x_j} - dx$  with  $p = 1$  and  $q = +\infty$ . By results in [2] and [13], there exist sequences of points with isotropic discrepancy, that is discrepancy of convex sets,

$$\sup_{\text{convex } A \subseteq [0,1]^d} \left\{ \left| N^{-1} \sum_{j=1}^N \chi_A(x_j) - |A| \right| \right\} \leq c N^{-2/(d+1)} \log^{\gamma}(N).$$

■

For polyhedra there are sequences with smaller discrepancy.

**Corollary 7** *Let  $\gamma = 1$  if  $d = 2$  and  $\gamma = d$  if  $d \geq 3$ . Then for every polyhedron  $\Omega$  contained in  $[0, 1]^d$  there is a constant  $c$  with the following property: For every  $N \geq 2$  there exists a finite sequence of points  $\{x_j\}_{j=1}^N$  in  $[0, 1]^d$  such that for every smooth function  $f$  on  $\mathbb{T}^d$ ,*

$$\begin{aligned} & \left| N^{-1} \sum_{j=1}^N (f \chi_{\Omega})(x_j) - \int_{\Omega} f(x) dx \right| \\ & \leq c N^{-1} \log^{\gamma}(N) \sum_{\alpha \in \{0,1\}^d} \int_{[0,1]^d} \left| \left( \frac{\partial}{\partial x} \right)^{\alpha} f(x) \right| dx. \end{aligned}$$

**Proof.** There exists a finite number of directions such that for every interval  $I$  with edges parallel to the axes, the facets of the polyhedra  $I \cap \Omega$  are perpendicular to these directions. Then one can apply Theorem 2.11 in [5] and deduce the existence of a finite sequence  $\{x_j\}_{j=1}^N$  with discrepancy

$$\sup_{I \subseteq [0,1]^d} \left\{ \left| N^{-1} \sum_{j=1}^N \chi_{I \cap \Omega}(x_j) - |I \cap \Omega| \right| \right\} \leq c N^{-1} \log^d(N).$$



When  $d = 2$ , Theorem 1 in [4] gives the better estimate  $cN^{-1} \log(N)$ . ■

The following is another analog of Theorem 1, with a larger variation but a smaller discrepancy.

**Theorem 8** *If  $0 < a < 1$  is an irrational number and if there exist  $\delta > 0$  and  $\gamma \geq 2$  with the property that  $|a - h/k| \geq \delta k^{-\gamma}$  for every rational  $h/k$ , then there exists a constant  $c > 0$ , which depends explicitly on  $\delta, \gamma, d$ , with the following property: If  $A = [-a/2, +a/2]^d$  is the cube centered at the origin with side length  $a$ , if  $\Omega$  is a Borel set in  $\mathbb{R}^d$ , if  $\mu$  is a periodic Borel measure, and if  $f$  is a periodic smooth function, then*

$$\begin{aligned} & \left| \int_{\Omega} f(x) \overline{d\mu(x)} \right| \\ & \leq c \left\{ \int_{\mathbb{T}^d} \left| \sum_{n \in \mathbb{Z}^d} \mu((x + n - A) \cap \Omega) \right|^2 dx \right\}^{1/2} \\ & \quad \times \left\{ \sum_{n \in \mathbb{Z}^d} \left( \prod_{k=1}^d (1 + |n_k|)^{2\gamma} \right) |\widehat{f}(n)|^2 \right\}^{1/2}. \end{aligned}$$

**Proof.** The periodization of  $\chi_A$  has Fourier expansion

$$g(x) = \sum_{n \in \mathbb{Z}^d} \left( \prod_{k=1}^d \frac{\sin(\pi a n_k)}{\pi n_k} \right) e^{2\pi i n \cdot x}.$$

By Lemma 2 applied to the periodization  $\nu$  of the measure  $\chi_{\Omega}\mu$ , with  $p = q = 2$ ,

$$\begin{aligned} & \left| \int_{\Omega} f(x) \overline{d\mu(x)} \right| \leq \left\{ \int_{\mathbb{T}^d} |g * \nu(x)|^2 dx \right\}^{1/2} \\ & \quad \times \left\{ \int_{\mathbb{T}^d} \left| \sum_{n \in \mathbb{Z}^d} \left( \prod_{k=1}^d \frac{\sin(\pi a n_k)}{\pi n_k} \right)^{-1} \widehat{f}(n) e^{2\pi i n \cdot x} \right|^2 dx \right\}^{1/2}. \end{aligned}$$

As in the proof of Lemma 5,

$$\left\{ \int_{\mathbb{T}^d} |g * \nu(x)|^2 dx \right\}^{1/2} = \left\{ \int_{\mathbb{T}^d} \left| \sum_{n \in \mathbb{Z}^d} \mu((x + n - A) \cap \Omega) \right|^2 dx \right\}^{1/2}.$$

By the assumptions,  $|\sin(\pi a n_k)| \geq c |n_k|^{1-\gamma}$  when  $n_k \neq 0$ . Then

$$\left| \prod_{k=1}^d \frac{\sin(\pi a n_k)}{\pi n_k} \right|^{-1} \leq c \prod_{k=1}^d (1 + |n_k|)^\gamma.$$

Hence, by Parseval equality,

$$\begin{aligned} & \left\{ \int_{\mathbb{T}^d} \left| \sum_{n \in \mathbb{Z}^d} \left( \prod_{k=1}^d \frac{\sin(\pi a n_k)}{\pi n_k} \right)^{-1} \widehat{f}(n) e^{2\pi i n \cdot x} \right|^2 dx \right\}^{1/2} \\ &= \left\{ \sum_{n \in \mathbb{Z}^d} \left| \left( \prod_{k=1}^d \frac{\sin(\pi a n_k)}{\pi n_k} \right)^{-1} \widehat{f}(n) \right|^2 \right\}^{1/2} \\ &\leq c \left\{ \sum_{n \in \mathbb{Z}^d} \left( \prod_{k=1}^d (1 + |n_k|)^{2\gamma} \right) |\widehat{f}(n)|^2 \right\}^{1/2}. \end{aligned}$$

■

In particular, if  $a$  is a quadratic irrational one can take  $\gamma = 2$  and  $\delta$  can be made explicit, and the variation of the function can be controlled by square norms of derivatives up to the order  $2d$ . This variation is larger than the one in the proof of Theorem 1, which is controlled by derivatives of order  $d$ . On the other hand, the discrepancy associated to the family of all intervals in Theorem 1 is larger than the discrepancy associated to the translated of a single cube in Corollary 8. Finally, there is an analog of the above theorem with balls instead of cubes. The zeros of Fourier transforms of characteristic functions of cubes play a crucial role in the above corollary. The Fourier transforms of balls can be expressed in terms of Bessel functions. The following lemma is about the zeros of Bessel functions.

**Lemma 9** *If  $J_\alpha$  is the Bessel function of first kind of order  $\alpha \geq -1/2$ , if  $\beta > 5/4$ , and if  $0 < a < b$ , then there exists  $c > 0$  and  $a < r < b$  with the property that for every positive integer  $k$ ,*

$$\left| J_\alpha(r\sqrt{k}) \right| \geq ck^{-\beta}.$$

**Proof.** The zeros of  $J_\alpha(t)$  are simple, with the possible exception of  $t = 0$ . If  $\varepsilon < 1$  then  $|J_\alpha(t)|^{-\varepsilon}$  is locally integrable in  $t > 0$  and, by the asymptotic

expansion of Bessel functions,

$$\begin{aligned} \int_a^b \left| \sqrt{r\sqrt{k}} J_\alpha(r\sqrt{k}) \right|^{-\varepsilon} dr &= k^{-1/2} \int_{a\sqrt{k}}^{b\sqrt{k}} \left| \sqrt{t} J_\alpha(t) \right|^{-\varepsilon} dt \\ &= k^{-1/2} \int_{a\sqrt{k}}^{b\sqrt{k}} \left| \sqrt{2/\pi} \cos(t - \alpha\pi/2 - \pi/4) + O(t^{-1}) \right|^{-\varepsilon} dt \leq c. \end{aligned}$$

Hence, if  $\varepsilon < 1$  and  $\eta > 1 + \varepsilon/4$ ,

$$\begin{aligned} &\int_a^b \left( \sum_{k=1}^{+\infty} k^{-\eta} \left| J_\alpha(r\sqrt{k}) \right|^{-\varepsilon} \right) dr \\ &\leq b^{\varepsilon/2} \sum_{k=1}^{+\infty} k^{\varepsilon/4 - \eta} \left( \int_a^b \left| \sqrt{r\sqrt{k}} J_\alpha(r\sqrt{k}) \right|^{-\varepsilon} dr \right) < +\infty. \end{aligned}$$

Since the series  $\sum_{k=1}^{+\infty} k^{-\eta} \left| J_\alpha(r\sqrt{k}) \right|^{-\varepsilon}$  converges for almost every  $r$ , for almost every  $r$  there exists  $c > 0$  such that

$$\left| J_\alpha(r\sqrt{k}) \right| \geq ck^{-\eta/\varepsilon}.$$

Finally observe that if the interval  $r\sqrt{k} \leq t \leq r\sqrt{k+1}$  contains a zero of  $J_\alpha(t)$  then  $\left| J_\alpha(r\sqrt{k}) \right| \leq ck^{-3/4}$ . Hence the thesis does not hold with  $\beta < 3/4$ . ■

**Theorem 10** *For every  $0 < a < b$  and  $\gamma > d/2 + 5/4$  there exist a constant  $c > 0$  and a radius  $a < r < b$  with the following properties: If  $B = \{|x| \leq r\}$  is the ball centered at the origin with radius  $r$ , if  $\Omega$  is a Borel set in  $\mathbb{R}^d$ , if  $\mu$  is a periodic Borel measure, and if  $f$  is a periodic smooth function, then*

$$\begin{aligned} &\left| \int_\Omega f(x) \overline{d\mu(x)} \right| \\ &\leq c \left\{ \int_{\mathbb{T}^d} \left| \sum_{n \in \mathbb{Z}^d} \mu((x+n-B) \cap \Omega) \right|^2 dx \right\}^{1/2} \left\{ \sum_{n \in \mathbb{Z}^d} (1+|n|^2)^\gamma \left| \widehat{f}(n) \right|^2 \right\}^{1/2}. \end{aligned}$$

**Proof.** The Fourier transform of the characteristic function of a ball is a Bessel function, and the periodization of  $\chi_B$  has Fourier expansion

$$g(x) = \sum_{n \in \mathbb{Z}^d} r^{d/2} |n|^{-d/2} J_{d/2}(2\pi r |n|) e^{2\pi i n \cdot x}.$$

By Lemma 2 applied to the periodization  $\nu$  of the measure  $\chi_\Omega \mu$ , with  $p = q = 2$ ,

$$\begin{aligned} \left| \int_\Omega f(x) \overline{d\mu(x)} \right| &\leq \left\{ \int_{\mathbb{T}^d} |g * \nu(x)|^2 dx \right\}^{1/2} \\ &\times \left\{ \int_{\mathbb{T}^d} \left| \sum_{n \in \mathbb{Z}^d} \left( r^{d/2} |n|^{-d/2} J_{d/2}(2\pi r |n|) \right)^{-1} \widehat{f}(n) e^{2\pi i n \cdot x} \right|^2 dx \right\}^{1/2}. \end{aligned}$$

As in the proof of Lemma 5,

$$\left\{ \int_{\mathbb{T}^d} |g * \nu(x)|^2 dx \right\}^{1/2} = \left\{ \int_{\mathbb{T}^d} \left| \sum_{n \in \mathbb{Z}^d} \mu((x + n - B) \cap \Omega) \right|^2 dx \right\}^{1/2}.$$

Moreover, by Parseval equality and Lemma 9, one can choose  $r$  so that

$$\begin{aligned} &\left\{ \int_{\mathbb{T}^d} \left| \sum_{n \in \mathbb{Z}^d} \left( r^{d/2} |n|^{-d/2} J_{d/2}(2\pi r |n|) \right)^{-1} \widehat{f}(n) e^{2\pi i n \cdot x} \right|^2 dx \right\}^{1/2} \\ &= \left\{ \sum_{n \in \mathbb{Z}^d} \left| \left( r^{d/2} |n|^{-d/2} J_{d/2}(2\pi r |n|) \right)^{-1} \widehat{f}(n) \right|^2 \right\}^{1/2} \\ &\leq c \left\{ \sum_{n \in \mathbb{Z}^d} (1 + |n|^2)^\gamma \left| \widehat{f}(n) \right|^2 \right\}^{1/2}. \end{aligned}$$

Finally observe that less than  $d/2$  square integrable derivatives are not enough to guarantee the boundedness of a functions. Hence the assumption  $\gamma > d/2 + 5/4$  is not too far to be best possible. ■

### 3 Koksma Hlawka inequalities on manifolds

The results in the previous section are of local nature and with a change of variables can be easily transferred from cubes to compact manifolds. Let  $\mathcal{M}$  be a smooth compact  $d$  dimensional manifold with a normalized measure  $dx$ . Fix a family of local charts  $\{\varphi_k\}_{k=1}^K$ ,  $\varphi_k : [0, 1]^d \rightarrow \mathcal{M}$ , and a smooth partition of unity  $\{\psi_k\}_{k=1}^K$  subordinate to these charts. The Sobolev spaces  $W^{n,p}(\mathcal{M})$  can be defined by the norms

$$\|f\|_{W^{n,p}(\mathcal{M})} = \sum_{1 \leq k \leq K} \sum_{|\alpha| \leq n} \left\{ \int_{[0,1]^d} \left| \frac{\partial^\alpha}{\partial x^\alpha} (\psi_k(\varphi_k(x)) f(\varphi_k(x))) \right|^p dx \right\}^{1/p}.$$

One can define an interval in  $\mathcal{M}$  as the image under a local chart of an interval in  $[0, 1]^d$ , say  $U = \varphi_k(I)$ . The discrepancy of a finite Borel measure  $\mu$  on  $\mathcal{M}$  with respect to the collection  $A$  of all intervals in  $\mathcal{M}$  is

$$\mathcal{D}(\mu) = \sup_{U \in A} \left| \int_U d\mu(y) \right|.$$

**Theorem 11** *There exists a constant  $c > 0$ , which depends on the local charts but not on the function  $f$  or the measure  $\mu$ , such that*

$$\left| \int_{\mathcal{M}} f(y) \overline{d\mu(y)} \right| \leq c \mathcal{D}(\mu) \|f\|_{W^{d,1}(\mathcal{M})}.$$

**Proof.** It suffices to prove the theorem for functions with support in the image of a single local chart  $\varphi : [0, 1]^d \rightarrow \mathcal{M}$ . If the measure  $\nu$  is the pull back on  $[0, 1]^d$  of the measure  $\mu$  on  $\mathcal{M}$  then, by Theorem 1,

$$\begin{aligned} \left| \int_{\mathcal{M}} f(y) \overline{d\mu(y)} \right| &= \left| \int_{[0,1]^d} f(\varphi(x)) \overline{d\nu(x)} \right| \\ &\leq 2^d \left\{ \sup_{I \subseteq [0,1]^d} \left| \int_{[0,1]^d} \chi_I(x) d\nu(x) \right| \right\} \\ &\times \left\{ \sum_{\alpha \in \{0,1\}^d} 2^{d-|\alpha|} \int_{[0,1]^d} \left| \left( \frac{\partial}{\partial x} \right)^\alpha f(\varphi(x)) \right| dx \right\}. \end{aligned}$$

The first factor is dominated by the discrepancy,

$$\sup_{I \subseteq [0,1]^d} \left\{ \left| \int_{[0,1]^d} \chi_I(x) d\nu(x) \right| \right\} \leq \sup_{U \in A} \left\{ \left| \int_{\mathcal{M}} \chi_U(y) d\mu(y) \right| \right\}.$$

The second factor is dominated by the Sobolev norm,

$$\sum_{\alpha \in \{0,1\}^d} 2^{d-|\alpha|} \int_{[0,1]^d} \left| \left( \frac{\partial}{\partial x} \right)^\alpha f(\varphi(x)) \right| dx \leq c \|f\|_{W^{d,1}(\mathcal{M})}.$$

■

For example, when the manifold is a 2 dimensional sphere and the local charts are central projection from a tangent plane to the sphere, the images of rectangles on the tangent plane are geodesic quadrilaterals on the sphere. Since a quadrilateral is union of two triangles, everything can be controlled by the discrepancy with respect to geodesic triangles. Finally, as in Theorem 10, one can consider a Koksma Hlawka inequality on the sphere, with

spherical cap discrepancy. The zonal polynomials  $Z_n(x \cdot y)$  on the sphere  $\mathcal{S} = \{x \in \mathbb{R}^3 : |x| = 1\}$  are the reproducing kernels of the spaces of harmonic polynomials of degree  $n$ . If  $Q_n(x)$  is a harmonic polynomial of degree  $n$ , then

$$Q_n(x) = \int_{\mathcal{S}} Z_n(x \cdot y) Q_n(y) dy.$$

Every distribution  $f$  on the sphere has a spherical harmonic expansion

$$f(x) = \sum_{n=0}^{+\infty} \int_{\mathcal{S}} Z_n(x \cdot y) f(y) dy = \sum_{n=0}^{+\infty} \widehat{f}(n, x).$$

The series converges in the topology of distributions. If  $f$  is square integrable, the series converges in the square norm, and if it is smooth, it also converges absolutely and uniformly. The following is a spherical analog of Lemma 2.

**Lemma 12** *Let  $f$  be an integrable function and  $\mu$  a finite measure on the sphere. Also let  $\varphi(n)$  be a non vanishing complex sequence on  $\mathbb{N}$ , and assume that both  $\varphi(n)$  and  $1/\varphi(n)$  have tempered growth. Define*

$$\begin{aligned} \mathfrak{D}f(x) &= \sum_{n=0}^{+\infty} \varphi(n) \widehat{f}(n, x), \\ g(x \cdot y) &= \sum_{n=0}^{+\infty} \overline{\varphi(n)^{-1}} Z_n(x \cdot y). \end{aligned}$$

Then

$$\left| \int_{\mathcal{S}} f(x) \overline{d\mu(x)} \right| \leq \left\{ \int_{\mathcal{S}} \left| \int_{\mathcal{S}} g(x \cdot y) d\mu(y) \right|^2 dx \right\}^{1/2} \left\{ \int_{\mathcal{S}} |\mathfrak{D}f(x)|^2 dx \right\}^{1/2}.$$

**Proof.** By the spherical harmonic expansions of  $f$  and  $\mu$ ,

$$\begin{aligned} & \left| \int_{\mathcal{S}} f(x) \overline{d\mu(x)} \right| \\ &= \left| \sum_{n=0}^{+\infty} \int_{\mathcal{S}} \widehat{f}(n, x) \overline{\widehat{\mu}(n, x)} dx \right| \\ &\leq \left\{ \int_{\mathcal{S}} \left| \sum_{n=0}^{+\infty} \varphi(n) \widehat{f}(n, x) \right|^2 dx \right\}^{1/2} \\ &\quad \times \left\{ \int_{\mathcal{S}} \left| \int_{\mathcal{S}} \left( \sum_{n=0}^{+\infty} \overline{\varphi(n)^{-1}} Z_n(x \cdot y) \right) d\mu(y) \right|^2 dx \right\}^{1/2}. \end{aligned}$$

■

In what follows two specific examples of sequences  $\varphi$  and functions  $g$  are considered. The following is an analog of Lemma 9, with Legendre polynomials in place of Bessel functions.

**Lemma 13** (1) *Let*

$$\chi_{\{x \cdot y \geq \cos(\vartheta)\}}(x \cdot y) = \sum_{n=0}^{+\infty} \overline{\varphi(n)^{-1}} Z_n(x \cdot y)$$

*be the spherical harmonic expansion of the characteristic function of the spherical cap  $\{x \cdot y \geq \cos(\vartheta)\}$  on the two dimensional sphere. Then for every  $\gamma > 5/2$  and for almost every  $0 < \vartheta < \pi$  there exists positive constants  $c_1$  and  $c_2$  such that for every positive integer  $n$ ,*

$$c_1 n^{3/2} \leq |\varphi(n)| \leq c_2 n^\gamma.$$

(2) *Let*

$$\chi_{\{x \cdot y \geq \cos(\vartheta)\}}(x \cdot y) + i \chi_{\{x \cdot y \geq \cos(2\vartheta)\}}(x \cdot y) = \sum_{n=0}^{+\infty} \overline{\varphi(n)^{-1}} Z_n(x \cdot y).$$

*Then for almost every  $0 < \vartheta < \pi/2$  there exist positive constants  $c_1$  and  $c_2$  such that for every positive integer  $n$ ,*

$$c_1 n^{3/2} \leq |\varphi(n)| \leq c_2 n^{3/2}.$$

**Proof.** The zonal polynomials on the two dimensional sphere are multiple of Legendre polynomials,

$$\begin{aligned} Z_n(x \cdot y) &= (2n+1) P_n(x \cdot y), \\ P_n(z) &= \frac{d^n (z^2 - 1)^n}{dz^n 2^n n!}. \end{aligned}$$

The characteristic function of the spherical cap  $\{x \cdot y \geq \cos(\vartheta)\}$  has the expansion

$$\begin{aligned} &\chi_{\{x \cdot y \geq \cos(\vartheta)\}}(x \cdot y) \\ &= \sum_{n=0}^{+\infty} \left( (n+1/2) \int_{\cos(\vartheta)}^1 P_n(z) dz \right) P_n(x \cdot y) \\ &= \frac{1 - \cos(\vartheta)}{2} + \sum_{n=1}^{+\infty} \frac{P_{n-1}(\cos(\vartheta)) - P_{n+1}(\cos(\vartheta))}{2} P_n(x \cdot y) \\ &= \frac{1 - \cos(\vartheta)}{2} Z_0(x \cdot y) + \sum_{n=1}^{+\infty} \frac{P_{n-1}(\cos(\vartheta)) - P_{n+1}(\cos(\vartheta))}{2(2n+1)} Z_n(x \cdot y). \end{aligned}$$

This follows from the identities

$$Z_n(x \cdot y) = (2n+1) P_n(z) = P'_{n+1}(z) - P'_{n-1}(z).$$

The Legendre polynomials in  $0 < a < \vartheta < b < \pi$  have the asymptotic expansion

$$P_n(\cos(\vartheta)) = \sqrt{\frac{2}{\pi n \sin(\vartheta)}} \cos((n+1/2)\vartheta + \pi/4) + O(n^{-3/2}).$$

Hence

$$\begin{aligned} & P_{n-1}(\cos(\vartheta)) - P_{n+1}(\cos(\vartheta)) \\ &= \sqrt{\frac{2}{\pi(n-1)\sin(\vartheta)}} \cos((n-1/2)\vartheta + \pi/4) \\ &\quad - \sqrt{\frac{2}{\pi(n+1)\sin(\vartheta)}} \cos((n+3/2)\vartheta + \pi/4) + O(n^{-3/2}) \\ &= \sqrt{\frac{2}{\pi n \sin(\vartheta)}} (\cos((n-1/2)\vartheta + \pi/4) - \cos((n+3/2)\vartheta + \pi/4)) + O(n^{-3/2}) \\ &= \sqrt{\frac{2}{\pi n \sin(\vartheta)}} 2 \sin(\vartheta) \sin((n+1/2)\vartheta + \pi/4) + O(n^{-3/2}). \end{aligned}$$

It follows from these estimates that  $|\varphi(n)| \geq cn^{3/2}$ . In order to prove a reverse inequality, observe that the above polynomial vanishes only when  $(n+1/2)\vartheta + \pi/4$  is close to a multiple of  $\pi$ , but at these points the derivative is large,

$$\begin{aligned} & \frac{d}{d\vartheta} (P_{n-1}(\cos(\vartheta)) - P_{n+1}(\cos(\vartheta))) \\ &= (2n+1) \sin(\vartheta) P_n(\cos(\vartheta)) \\ &= (2n+1) \sqrt{\frac{2 \sin(\vartheta)}{\pi n}} \cos((n+1/2)\vartheta + \pi/4) + O(n^{-1/2}). \end{aligned}$$

In particular, if  $n$  is large, say  $n \geq N$ , and  $0 < a < \vartheta < b < \pi$  then the zeros are simple. This implies that in (1),

$$\begin{aligned} \varphi(n) &= \left( \frac{P_{n-1}(\cos(\vartheta)) - P_{n+1}(\cos(\vartheta))}{2(2n+1)} \right)^{-1} \\ &= \frac{\sqrt{\pi n} (2n+1)}{\sqrt{2 \sin(\vartheta)} (\sin((n+1/2)\vartheta + \pi/4) + O(1/n))}. \end{aligned}$$



Finally, as in the proof of Lemma 9, if  $\varepsilon < 1$  and  $\eta > 1 + 3\varepsilon/2$  then

$$\int_a^b \left( \sum_{n=N}^{+\infty} n^{-\eta} |\varphi(n)|^\varepsilon \right) d\vartheta < +\infty.$$

If the series  $\sum_{n=1}^{+\infty} n^{-\eta} |\varphi(n)|^\varepsilon$  converges for almost every  $\vartheta$ , then for almost every  $\vartheta$  there exists  $c > 0$  such that

$$|\varphi(n)| \leq cn^{\eta/\varepsilon}.$$

This proves (1). The proof of (2) is a bit different. Let

$$\chi_{\{x \cdot y \geq \cos(\vartheta)\}}(x \cdot y) + i \chi_{\{x \cdot y \geq \cos(2\vartheta)\}}(x \cdot y) = \sum_{n=0}^{+\infty} \overline{\varphi(n)^{-1}} Z_n(x \cdot y).$$

Then, if  $n = 1, 2, 3, \dots$ ,

$$\begin{aligned} & |\varphi(n)| \\ &= \left| \frac{P_{n-1}(\cos(\vartheta)) - P_{n+1}(\cos(\vartheta))}{2(2n+1)} + i \frac{P_{n-1}(\cos(2\vartheta)) - P_{n+1}(\cos(2\vartheta))}{2(2n+1)} \right|^{-1} \\ &= \left( \left| \frac{P_{n-1}(\cos(\vartheta)) - P_{n+1}(\cos(\vartheta))}{2(2n+1)} \right|^2 + \left| \frac{P_{n-1}(\cos(2\vartheta)) - P_{n+1}(\cos(2\vartheta))}{2(2n+1)} \right|^2 \right)^{-1/2} \\ &= \sqrt{\pi n/2} (2n+1) (|\sin(\vartheta)| \sin^2((n+1/2)\vartheta + \pi/4) \\ &\quad + |\sin(2\vartheta)| \sin^2((2n+1)\vartheta + \pi/4) + O(1/n))^{-1/2}. \end{aligned}$$

If  $0 < a < \vartheta < b < \pi/2$  and if  $n$  is large, then

$$\begin{aligned} & |\sin(\vartheta)| \sin^2((n+1/2)\vartheta + \pi/4) + |\sin(2\vartheta)| \sin^2((2n+1)\vartheta + \pi/4) + O(1/n) \\ & \geq c (\sin^2(\omega + \pi/4) + \sin^2(2\omega + \pi/4)) + O(1/n) \\ & \geq c + O(1/n) \geq c > 0. \end{aligned}$$

In particular, there exists  $N$  such that for every  $n \geq N$  and every  $0 < a < \vartheta < b < \pi/2$ , one has  $|\varphi(n)| \leq cn^{3/2}$ . Moreover, the equations  $\varphi(n) = 0$  for some  $n < N$  have a finite number of solutions  $0 < a < \vartheta < b < \pi/2$ . Hence, if  $\vartheta$  is not one of these solutions, then it satisfies (2). ■

**Theorem 14** (1) For every  $\gamma > 5/2$  and almost every  $0 < \vartheta < \pi$  there exists a constant  $c > 0$  with the following property: If  $B(x, \vartheta) = \{x \cdot y \geq \cos(\vartheta)\}$

are the spherical caps with center  $x$  and radius  $\vartheta$ , if  $\Omega$  is a Borel set, if  $\mu$  is a Borel measure, and if  $f$  is a smooth function in  $\mathcal{S}$ , then

$$\begin{aligned} & \left| \int_{\Omega} f(x) \overline{d\mu(x)} \right| \\ & \leq c \left\{ \int_{\mathcal{S}} |\mu(B(x, \vartheta) \cap \Omega)|^2 dx \right\}^{1/2} \left\{ \sum_{n=0}^{+\infty} (1+n^2)^{\gamma} \int_{\mathcal{S}} |\widehat{f}(n, x)|^2 dx \right\}^{1/2}. \end{aligned}$$

(2) For almost every radius  $0 < \vartheta < \pi/2$  there exists a constant  $c > 0$  with the property that

$$\begin{aligned} & \left| \int_{\Omega} f(x) \overline{d\mu(x)} \right| \\ & \leq c \left\{ \sum_{n=0}^{+\infty} (1+n^2)^{3/2} \int_{\mathcal{S}} |\widehat{f}(n, x)|^2 dx \right\}^{1/2} \\ & \times \left( \left\{ \int_{\mathcal{S}} |\mu(B(x, \vartheta) \cap \Omega)|^2 dx \right\}^{1/2} + \left\{ \int_{\mathcal{S}} |\mu(B(x, 2\vartheta) \cap \Omega)|^2 dx \right\}^{1/2} \right). \end{aligned}$$

**Proof.** If  $g(x \cdot y) = \chi_{\{x \cdot y \geq \cos(\vartheta)\}}(x \cdot y)$  and if  $\nu$  is the restriction of the measure  $\mu$  to the set  $\Omega$ , then

$$\left\{ \int_{\mathcal{S}} \left| \int_{\mathcal{S}} g(x \cdot y) d\nu(y) \right|^2 dx \right\}^{1/2} = \left\{ \int_{\mathcal{S}} |\mu(B(x, \vartheta) \cap \Omega)|^2 dx \right\}^{1/2}.$$

Then (1) follows from Lemma 12 and Lemma 13 (1). If  $g(x \cdot y) = \chi_{\{x \cdot y \geq \cos(\vartheta)\}}(x \cdot y) + i\chi_{\{x \cdot y \geq \cos(2\vartheta)\}}(x \cdot y)$  and if  $\nu$  is the restriction of the measure  $\mu$  to the set  $\Omega$ , then

$$\begin{aligned} & \left\{ \int_{\mathcal{S}} \left| \int_{\mathcal{S}} g(x \cdot y) d\nu(y) \right|^2 dx \right\}^{1/2} \\ & \leq \left\{ \int_{\mathcal{S}} \left| \int_{\mathcal{S}} \chi_{\{x \cdot y \geq \cos(\vartheta)\}}(x \cdot y) d\nu(y) \right|^2 dx \right\}^{1/2} \\ & \quad + \left\{ \int_{\mathcal{S}} \left| \int_{\mathcal{S}} \chi_{\{x \cdot y \geq \cos(2\vartheta)\}}(x \cdot y) d\nu(y) \right|^2 dx \right\}^{1/2} \\ & = \left\{ \int_{\mathcal{S}} |\mu(B(x, \vartheta) \cap \Omega)|^2 dx \right\}^{1/2} + \left\{ \int_{\mathcal{S}} |\mu(B(x, 2\vartheta) \cap \Omega)|^2 dx \right\}^{1/2}. \end{aligned}$$

Then, as before, (2) follows from Lemma 12 and Lemma 13 (2). ■

Observe that the indices  $\gamma > 9/4$  in Theorem 10 with  $d = 2$  and  $\gamma > 5/2$  in Theorem 14 (1) are different. Anyhow, it is likely that both indices are not best possible. Finally, an analog of Theorem 14 (1) holds on spheres of dimension  $d > 2$  with  $\gamma > (d + 3)/2$ .

## 4 An application

In order to test the quality of the above results, we reconsider an example in [7]. Let

$$f(x_1, x_2, \dots, x_d) = \frac{1}{x_1 x_2 \cdots x_d (1 - x_1 - x_2 - \dots - x_d)}.$$

Also, for  $\varepsilon > 0$  small, let  $\Sigma$  be the simplex

$$\Sigma = \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_1 \geq \dots \geq x_d \geq \varepsilon, 1 - x_1 - \dots - x_d \geq \varepsilon\}.$$

One can show that

$$\sum_{|\alpha| \leq d} \int_{\Sigma} \left| \left( \frac{\partial}{\partial x} \right)^{\alpha} f(x) \right| dx \leq c\varepsilon^{-d}.$$

By Corollary 6 there exists a finite sequence  $\{x_j\}_{j=1}^N$  in  $[0, 1]^d$  such that for all convex sets  $\Omega$  contained in  $\Sigma$ ,

$$\left| N^{-1} \sum_{j=1}^N (f\chi_{\Omega})(x_j) - \int_{\Omega} f(x) dx \right| \leq c\varepsilon^{-d} N^{-2/(d+1)} \log^{\gamma}(N).$$

This agrees with the result in [7]. However, in the case  $\Omega = \Sigma$ , Corollary 6 gives the better estimate

$$\left| N^{-1} \sum_{j=1}^N (f\chi_{\Sigma})(x_j) - \int_{\Sigma} f(x) dx \right| \leq c\varepsilon^{-d} N^{-1} \log^d(N).$$

## References

- [1] C. Amstler, P. Zinterhof, *Uniform distribution, discrepancy, and reproducing kernel Hilbert spaces*, Journal Complexity 17 (2001), 497-515.
- [2] J. Beck, *On the discrepancy of convex plane sets*, Monatshefte Mathematik 105 (1988), 91-106.

- [3] J. Beck, W.W.L. Chen, *Irregularities of Distribution*, Cambridge University Press 2008.
- [4] W.W.L. Chen, G. Travaglini, *Discrepancy with respect to convex polygons*, Journal Complexity 23 (2007), 662-672.
- [5] L. Colzani, G. Gigante, G. Travaglini, *Trigonometric approximation and a general form of the Erdős-Turan inequality*, Transactions American Mathematical Society 363 (2011), 1101-1123.
- [6] G. Harman, *Metric number theory*, Oxford Science Publications 2007.
- [7] G. Harman, *Variations on the Koksma-Hlawka inequality*, Uniform Distribution Theory 5 (2010), 65-78.
- [8] F.G. Hickenell, *Koksma-Hlawka inequality*, in Encyclopedia of Statistical Sciences, Wiley 2006.
- [9] E. Hlawka, *The Theory of Uniform Distribution*, A B Academic Publisher 1984.
- [10] L. Kuipers, H. Niederreiter, *Uniform Distribution of Sequences*, Dover 2002.
- [11] J. Matoušek, *Geometric Discrepancy*, Springer 2010.
- [12] H. Niederreiter, *Random Number Generation and Quasi-Monte Carlo Methods*, SIAM 1992.
- [13] W. Stute, *Convergence rates for the isotrope discrepancy*, Annals Probability 5 (1977), 707-723.
- [14] G. Travaglini, *Appunti su Teoria dei numeri, Analisi di Fourier e Distribuzione di Punti*, UMI-Pitagora (2010).
- [15] S.K. Zaremba, *Some applications of multidimensional integration by parts*, Annales Polonici Mathematici 21 (1968), 85-96.

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